

MORAVA K -THEORY OF TWISTED FLAG VARIETIES

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ABSTRACT. In the present article we prove some results about the Morava K -theory. In particular, we construct an operation from the Morava K -theory to the Chow theory analogous to the second Chern class for Grothendieck's K^0 -theory. Furthermore, we investigate ordinary and equivariant oriented cohomology theories in the sense of Levine-Morel of projective quadrics, and discuss the Rost motives.

1. INTRODUCTION

The concept of oriented cohomology theories is well-known in *algebraic topology*. Levine and Morel introduced in [LM] a universal oriented cohomology theory in *algebraic geometry* — the algebraic cobordism. Using it one can define different other cohomology theories by the change of coefficients.

Namely, like in topology every oriented cohomology theory is equipped with a formal group law. For example, for the Chow theory this is the additive formal groups law and for Grothendieck's K^0 -theory this is the multiplicative formal group law. Further, taking the Lubin-Tate formal group law one can define the Morava K -theories in algebraic geometry. We remark that in algebraic topology the Morava K -theory is a well-established theory (see [Rav]).

Due to works of Levine, Morel, Panin and Smirnov there exist the Riemann-Roch-type theorems which can be used to investigate different aspects of oriented cohomology theories. Moreover, recently an equivariant version of algebraic cobordism has appeared and, hence, one can define arbitrary equivariant oriented theories in algebraic geometry.

The algebraic cobordism theory has found applications in the theory of linear algebraic groups and quadratic forms mainly due to the works of Vishik (e.g. in the construction of fields with the u -invariant of the form $2^r + 1$, $r \geq 3$). Moreover, the classical theories, like the Chow theory and K^0 are known to give strong applications in the classification of central simple algebras, quadratic forms and algebraic groups (e.g. the index reduction formulae of Merkurjev, Panin, Wadsworth; diverse results on quadratic forms due to Karpenko and Vishik — see e.g. the introduction in [GPS14]).

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In the present article we investigate the Morava K -theory of algebraic varieties and provide a tool to compute the motive with respect to an arbitrary oriented cohomology theory of projective quadrics. We start in Section 2 summarizing some general results related to oriented cohomology theories. Section 3 is devoted to a construction of a surjective operation from the Morava K -theory to the Chow theory analogous to the second Chern class for K^0 . This operation turns out to be additive when we consider it as an operation from the Morava K -theory to the Witt group of the Chow ring associated with the Lubin-Tate formal group law.

In Section 4 we discuss the Morava-Rost motives and in the remaining sections we introduce an algorithm to compute the motive of a projective quadric with respect to an arbitrary oriented cohomology theory. This algorithm essentially involves the equivariant theories.

2. DEFINITIONS AND NOTATION

We follow [Ful], [LM], [Pa03], [Haz], [Rav].

2.1 (Oriented cohomology theories and motives). Let F be a field of characteristic 0. We denote by A^* an oriented cohomology theory in the sense of Levine-Morel over F which satisfies the localization axiom and is generically constant ([LM, Def. 4.4.1, Def. 4.4.6], cf. [Pa03], [Sm06]). In particular, we consider $A^* = \text{CH}^*$ the Chow ring modulo rational equivalence, $A^* = K^0[v_1, v_1^{-1}]$ the Grothendieck group of locally free coherent sheaves, and $A^* = \Omega^*$ the algebraic cobordism.

By [LM, Thm. 1.2.6] the algebraic cobordism is a universal oriented cohomology theory and there is a (unique) morphism of theories $\Omega^* \rightarrow A^*$ for any oriented cohomology theory A^* .

Each oriented cohomology theory A is equipped with a 1-dimensional commutative formal group law FGL_A . For CH^* this is the additive formal group law, for K^0 the multiplicative formal group law and for Ω the universal formal group law. Moreover, these theories are universal for the respective formal group laws.

For a theory A^* we consider the category of A^* -motives with coefficients in a commutative ring R , which is defined in the same way as the category of Grothendieck's Chow motives with CH^* replaced by $A^* \otimes_{\mathbb{Z}} R$ (see [Ma68]). In the present article the ring R is \mathbb{Z} , $\mathbb{Z}_{(p)}$, or \mathbb{F}_p for a prime number p .

2.2 (Morava-like theories). For a prime number p and a natural number n we consider the n -th Morava K -theory $K(n)$ with respect to p . Note that we do not include p in the notation. We define this theory as the universal oriented cohomology theory for the Lubin-Tate formal group law of height n with the coefficient ring $\mathbb{Z}_{(p)}[v_n, v_n^{-1}]$ (see Section 2.3).

For a variety X over F one has

$$K(n)(X) = \Omega(X) \otimes_{\mathbb{L}} \mathbb{Z}_{(p)}[v_n, v_n^{-1}],$$

and v_n is a ν_n -element in the Lazard ring \mathbb{L} (see e.g. [Sem13, Def. 2.3]). The degree of v_n is negative and equals $-(p^n - 1)$. In particular, $K(n)(\text{Spec } F) = \mathbb{Z}_{(p)}[v_n, v_n^{-1}]$. We remark that usually one considers the Morava K -theory with the coefficient ring $\mathbb{F}_p[v_n, v_n^{-1}]$.

If $n = 1$ and $p = 2$, one has $K(1)(X) = K^0(X)[v_1, v_1^{-1}] \otimes \mathbb{Z}_{(2)}$, since the Lubin-Tate formal group law is isomorphic to the multiplicative formal group law in this case.

Moreover, sometimes we consider the Brown-Peterson cohomology BP with coefficient ring $\mathbb{Z}_{(p)}[v_1, v_2, \dots]$. The n -th Morava K -theory can be obtained from BP by sending all v_i with $i \neq n$ to zero and localizing at v_n . Besides this, we consider the connective Morava K -theory $CK(n)$, where we do not invert v_n . The coefficient ring of $CK(n)$ equals, thus, $\mathbb{Z}_{(p)}[v_n]$.

2.3 (Lubin-Tate formal group law). We construct the formal group law for the n -th Morava K -theory modulo p following [Haz]. The logarithm of the formal group law of the Brown-Peterson cohomology equals

$$l(t) = \sum_{i \geq 0} m_i t^{p^i},$$

where $m_0 = 1$ and the remaining variables m_i are related to v_j as follows:

$$m_j = \frac{1}{p} \cdot \left(v_j + \sum_{i=1}^{j-1} m_i v_{j-i}^{p^i} \right).$$

Let $e(t)$ be the compositional inverse of $l(t)$. The Brown-Peterson formal group law is given by $e(l(x) + l(y))$.

The n -th Morava formal group law is obtained from the BP formal group law by sending all v_j with $j \neq n$ to zero. Modulo the ideal I generated by p, x^{p^n}, y^{p^n} the formal group law for the n -th Morava K -theory equals

$$\text{FGL}_{K(n)}(x, y) = x + y - v_n \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} x^{ip^{n-1}} y^{(p-i)p^{n-1}} \mod I.$$

In the same way, the logarithm of $K(n)$ is obtained from $l(t)$ by sending all v_j with $j \neq n$ to zero.

2.4 (Generalized Witt vectors). Let S be a commutative ring of characteristics 0 and

$$l(x) = \sum_{i=1}^{\infty} a_i x^i \in S \otimes \mathbb{Q}[[x]], \quad a_1 = 1$$

the logarithm of a formal group law over S . Following [Haz, 15.3] we define an abelian group of Witt vectors associated with $l(x)$ as follows.

First we define polynomials $w_n(z)$ in z_1, \dots, z_n as

$$w_n(z) = \sum_{d|n} a_{n/d} z_d^{n/d}.$$

Next define the polynomials $\Sigma_n(x_1, \dots, x_n; y_1, \dots, y_n)$ as

$$w_n(\Sigma_1(x; y), \dots, \Sigma_n(x; y)) = w_n(x) + w_n(y).$$

Then the coefficients of Σ_i lie in S (not just in $S \otimes \mathbb{Q}$).

Consider the set $W(S)$ of vectors of infinite length with coordinates in S and define the structure of an abelian group on $W(S)$ by

$$(a_1, a_2, \dots) +_{W(S)} (b_1, b_2, \dots) = (\Sigma_1(a; b), \Sigma_2(a; b), \dots)$$

The group $W(S)$ is the group of generalized Witt vectors. By construction it depends on $l(x)$. Moreover, note that the addition in $W(S)$ involves the multiplication in the ring S . The classical Witt vectors is a particular case of this general construction, when one starts with the multiplicative formal group law.

Example 2.5. Consider the logarithm $l(x)$ of the Morava K -theory mod p and send additionally v_n to 1. A straightforward computation shows that

$$l(x) = x + \frac{1}{p} \cdot x^{p^n} + \mathcal{O}(x^{2p^{n-1}}),$$

where \mathcal{O} is the Landau symbol. Then

$$w_1(z_1) = z_1, w_2(z_1, z_2) = z_2, \dots, w_{p^n-1}(z_1, \dots, z_{p^n-1}) = z_{p^n-1},$$

$$w_{p^n}(z_1, \dots, z_{p^n}) = z_{p^n} + \frac{1}{p} \cdot z_1^{p^n},$$

$$\Sigma_1(x; y) = x_1 + y_1, \Sigma_2(x; y) = x_2 + y_2, \dots, \Sigma_{p^n-1}(x; y) = x_{p^n-1} + y_{p^n-1},$$

$$\Sigma_{p^n}(x; y) = x_{p^n} + y_{p^n} + \frac{1}{p}(x_1^{p^n} + y_1^{p^n} - (x_1 + y_1)^{p^n}).$$

2.6 (Riemann-Roch theorem). Let A be an oriented cohomology theory, which is universal for its formal group law FGL_A .

Consider a morphism

$$\varphi: \text{FGL}_{\text{CH}_{\mathbb{Q}}} \rightarrow \text{FGL}_A$$

of formal group laws, i.e., $\text{FGL}_A(\varphi(x), \varphi(y)) = \varphi(x + y)$. We always choose the power series $\varphi(t)$ in such a way that its coefficient at t equals 1. This morphism induces a morphism of oriented cohomology theories

$$\mathbf{c}_\varphi: A^* \rightarrow \text{CH}^* \otimes A^*(F)_{\mathbb{Q}}.$$

Let E be a vector bundle over a smooth variety X and let α_i be its roots. Then the Todd class is defined as $\text{td}_\varphi(E) = \prod \text{td}_\varphi(\alpha_i)$ and $\text{td}_\varphi(t) = \frac{t}{\varphi(t)}$. The inverse Todd class

of E is defined as $\text{itd}_\varphi(E) = \prod \text{itd}_\varphi(\alpha_i)$ and $\text{itd}_\varphi(t) = \frac{\varphi(t)}{t}$.

For the n -th Morava K -theory the function

$$\varphi: \text{FGL}_{\text{CH}} \rightarrow \text{FGL}_{K(n)}$$

is obtained from $e(t)$ of Section 2.3 by sending all v_j with $j \neq n$ to zero.

The following proposition and its corollary is the Riemann-Roch theorem for general oriented cohomology theory developed by Levine, Morel, Panin and Smirnov in [Sm06], [Pa03, Thm. 2.5.3, 2.5.4], [LM, Ch. 4].

Proposition 2.7. *Let A be an oriented cohomology theory, which is universal for its formal group law.*

- (1) Let $i: Y \rightarrow X$ be a closed embedding of smooth varieties with the normal bundle N over Y and $\alpha \in A(Y)$. Then

$$i_*(\mathbf{c}_\varphi(\alpha) \cdot \mathrm{itd}_\varphi(N)) = \mathbf{c}_\varphi(i_*(\alpha)).$$

- (2) Let $f: Y \rightarrow X$ be a morphism of smooth projective varieties and $\alpha \in A(Y)$. Then

$$f_*(\mathbf{c}_\varphi(\alpha) \cdot \mathrm{td}_\varphi(T_Y)) = \mathbf{c}_\varphi(f_*(\alpha)) \cdot \mathrm{td}_\varphi(T_X),$$

where T_X and T_Y denote the tangent bundle over X and Y .

Corollary 2.8. Let X be a smooth projective variety, $\pi: X \rightarrow \mathrm{Spec} F$ the structural morphism, and $\alpha \in A^*(X)$. Then

$$\deg(\mathbf{c}_\varphi(\alpha) \cdot \mathrm{td}_\varphi(T_X)) = \mathbf{c}_\varphi(\pi_*(\alpha)).$$

For the Chern class $c_1^A(L) \in A^1(X)$ of a line bundle L over X we have

$$\mathbf{c}_\varphi(c_1^A(L)) = \varphi(c_1^{\mathrm{CH}}(L)).$$

Proof. We illustrate the method proving the last identity with the first Chern class of L . Let $s: X \rightarrow L$ be the zero section of the line bundle L . Then $c_1^A(L) = s_A^* s_*^A(1_X)$, where s_A^* and s_*^A denote the pullback and the pushforward for the theory A .

Then $\mathbf{c}_\varphi(c_1^A(L)) = \mathbf{c}_\varphi(s_A^* s_*^A(1_X)) = s_{\mathrm{CH}}^* \mathbf{c}_\varphi(s_*^A(1_X)) = s_{\mathrm{CH}}^* s_*^{\mathrm{CH}}(\mathrm{itd}_\varphi(N))$, where N is the normal line bundle to the zero section. By [Ful, Ex. 2.5.5] this bundle is the restriction of $\mathcal{O}_L(X)$ to X and $s_*^{\mathrm{CH}}(1) = [X] = c_1^{\mathrm{CH}}(\mathcal{O}_L(X))$.

Write $\varphi(t) = \sum_{i \geq 1} a_i t^i$ as a power series. Then

$$\begin{aligned} \mathrm{itd}_\varphi(N) &= \mathrm{itd}_\varphi(c_1^{\mathrm{CH}}(N)) = \sum_{i \geq 1} a_i (c_1^{\mathrm{CH}}(N))^{i-1} = \sum_{i \geq 1} a_i (c_1^{\mathrm{CH}}(s^* \mathcal{O}_L(X)))^{i-1} \\ &= \sum_{i \geq 1} a_i (s_{\mathrm{CH}}^* (c_1^{\mathrm{CH}}(\mathcal{O}_L(X))))^{i-1} = \sum_{i \geq 1} a_i s_{\mathrm{CH}}^* ((c_1^{\mathrm{CH}}(\mathcal{O}_L(X)))^{i-1}). \end{aligned}$$

Therefore

$$\begin{aligned} s_{\mathrm{CH}}^* s_*^{\mathrm{CH}}(\mathrm{itd}_\varphi(N)) &= s_{\mathrm{CH}}^* s_*^{\mathrm{CH}} \left(\sum_{i \geq 1} a_i s_{\mathrm{CH}}^* ((c_1^{\mathrm{CH}}(\mathcal{O}_L(X)))^{i-1}) \right) \\ &= s_{\mathrm{CH}}^* \left(\sum_{i \geq 1} a_i s_*^{\mathrm{CH}} (s_{\mathrm{CH}}^* ((c_1^{\mathrm{CH}}(\mathcal{O}_L(X)))^{i-1})) \right) = s_{\mathrm{CH}}^* \left(\sum_{i \geq 1} a_i (c_1^{\mathrm{CH}}(\mathcal{O}_L(X)))^{i-1} s_*^{\mathrm{CH}}(1) \right) \\ &= s_{\mathrm{CH}}^* \left(\sum_{i \geq 1} a_i [X]^{i-1} s_*^{\mathrm{CH}}(1) \right) = s_{\mathrm{CH}}^* \left(\sum_{i \geq 1} a_i (s_*^{\mathrm{CH}}(1))^i \right) = \sum_{i \geq 1} a_i s_{\mathrm{CH}}^* ((s_*^{\mathrm{CH}}(1))^i) \\ &= \sum_{i \geq 1} a_i (s_{\mathrm{CH}}^* (s_*^{\mathrm{CH}}(1)))^i = \sum_{i \geq 1} a_i (c_1^{\mathrm{CH}}(L))^i = \varphi(c_1^{\mathrm{CH}}(L)), \end{aligned}$$

where we used the projection formula and the multiplicativity of pullbacks. \square

2.9 (Euler characteristic). The Euler characteristic of a smooth projective variety X with respect to an oriented cohomology theory A^* is defined as $\pi_*^A(1_X) \in A^*(\mathrm{Spec} F)$, where $\pi: X \rightarrow \mathrm{Spec} F$ is the structural map.

The Euler characteristic can be computed using the Riemann-Roch theorem. E.g., for $A = K^0[v_1, v_1^{-1}]$ the Euler characteristic of X equals

$$v_1^{\dim X} \cdot \sum (-1)^i \dim H^i(X, \mathcal{O}_X)$$

see [Ful, Ch. 15]. If X is geometrically cellular, then this element equals $v_1^{\dim X}$.

For the Morava K -theory $K(n)$ and a variety X of dimension $d = p^n - 1$ the Euler characteristic modulo p equals the element $v_n \cdot u \cdot s_d$ for some $u \in \mathbb{F}_p^\times$, where s_d is the *Milnor number* of X (see [LM, Sec. 4.4.4], [Sem13, Sec. 2.2]). If $\dim X$ is not divisible by $p^n - 1$, then the Euler characteristic of X equals zero modulo p (see [LM, Prop. 4.4.22]).

2.10 (Rost nilpotence for oriented cohomology theories). Let A be an oriented cohomology theory and consider the category of A -motives over F . For a smooth projective variety X over F we denote by $\mathcal{M}(X)$ its motive (A -motive). We say that the Rost nilpotence principle holds for X , if the kernel of the restriction homomorphism

$$\mathrm{End}(\mathcal{M}(X)) \rightarrow \mathrm{End}(\mathcal{M}(X_E))$$

consists of nilpotent correspondences for all field extensions E/F .

Usually Rost nilpotency is formulated for Chow motives. By [CGM05, Sec. 8] it holds for all twisted flag varieties. Note that the proof of [CGM05] works for A -motives of twisted flag varieties for all oriented cohomology theories A satisfying the localization property.

3. SOME OPERATIONS IN THE MORAVA K -THEORY

A straightforward computation shows the following lemma:

Lemma 3.1. *Let p be a prime number and n be an integer. Consider the power series $\varphi: \mathrm{FGL}_{\mathrm{CH}} \rightarrow \mathrm{FGL}_{K(n)}$ of Section 2.6. Then*

$$\varphi(t) = t - \frac{1}{p} \cdot v_n t^{p^n} + \mathcal{O}(t^{2p^n-1}),$$

where \mathcal{O} is the Landau symbol.

Let now $\mathfrak{c} := \mathfrak{c}_\varphi = \mathfrak{c}^{K(n)}: K(n)^* \rightarrow \mathrm{CH}^* \otimes \mathbb{Q}[v_n, v_n^{-1}]$ be the respective morphism of theories and \mathfrak{c}_i its codimension i component. We remark that if $n = 1$ and $p = 2$, then \mathfrak{c} is the classical Chern character from K^0 to $\mathrm{CH}_{\mathbb{Q}}$.

Following notation of Houton [Ha12] define

$$\mathrm{CH}_{\mathbb{Z}_{(p)} \subset \mathbb{Q}}[v_n, v_n^{-1}] = \mathrm{Im}(\mathrm{CH} \otimes_{\mathbb{Z}_{(p)}} [v_n, v_n^{-1}] \rightarrow \mathrm{CH} \otimes \mathbb{Q}[v_n, v_n^{-1}]).$$

Theorem 3.2. (1) *For $i \leq p^n - 1$ the image of \mathfrak{c}_i lies in $\mathrm{CH}_{\mathbb{Z}_{(p)} \subset \mathbb{Q}}[v_n, v_n^{-1}]$.*

(2) *The image of $\mathfrak{b} := \mathfrak{c}_{p^n} + \frac{1}{p} \cdot \mathfrak{c}_1^{p^n}$ also lies in $\mathrm{CH}_{\mathbb{Z}_{(p)} \subset \mathbb{Q}}[v_n, v_n^{-1}]$.*

(3) *For $i \leq p^n - 1$ the maps \mathfrak{c}_i are onto $\mathrm{CH}_{\mathbb{Z}_{(p)} \subset \mathbb{Q}}^i[v_n, v_n^{-1}]$ and the map \mathfrak{b} is onto $\mathrm{CH}_{\mathbb{Z}_{(p)} \subset \mathbb{Q}}^{p^n}[v_n, v_n^{-1}]$.*

(4) *The map $(\mathfrak{c}_1, \dots, \mathfrak{c}_{p^n-1}, -\mathfrak{b}, 0, \dots)$ is a group homomorphism*

$$K(n) \rightarrow W(\mathrm{CH}_{\mathbb{Z}_{(p)} \subset \mathbb{Q}}^*[v_n, v_n^{-1}]),$$

where W are the generalized Witt vectors associated with the Lubin-Tate formal group law of Example 2.5.

Proof. Let X be a smooth projective variety over F .

For each closed subvariety $Z \subset X$ let $\tilde{Z} \rightarrow Z$ be its resolution of singularities. By [LM, Thm. 4.4.7] $K(n)(X)$ is generated as a $K(n)(\text{pt})$ -module by the classes $[\tilde{Z} \rightarrow X]_{K(n)}$. The morphisms \mathbf{c}_i are additive and

$$\mathbf{b}(\alpha + \beta) = \mathbf{b}(\alpha) + \mathbf{b}(\beta) + \frac{1}{p}((\mathbf{c}_1(\alpha) + \mathbf{c}_1(\beta))^{p^n} - \mathbf{c}_1(\alpha)^{p^n} - \mathbf{c}_1(\beta)^{p^n}).$$

Moreover, \mathbf{c} is $\mathbb{Z}_{(p)}[v_n, v_n^{-1}]$ -linear. Therefore it is sufficient to prove the integrality of \mathbf{c}_i and \mathbf{b} on generators. We compute $\mathbf{c}_i([\tilde{Z} \rightarrow X]_{K(n)})$ next.

We have by Riemann-Roch

$$(3.3) \quad \mathbf{c}_i([\tilde{Z} \xrightarrow{f} X]_{K(n)}) = \mathbf{c}_i(f_*^{K(n)}(1_{\tilde{Z}})) \\ = \text{codimension } i \text{ component of } \text{td}_\varphi(-T_X) \cdot f_*(\text{td}_\varphi(T_{\tilde{Z}})).$$

Computing td_φ using Lemma 3.1, this implies for $i \leq p^n - 1$ that

$$\mathbf{c}_i([\tilde{Z} \rightarrow X]) = \begin{cases} [Z], & \text{if } \text{codim } Z = i; \\ 0, & \text{otherwise.} \end{cases}$$

This proves the first three statements of the theorem for $i \leq p^n - 1$.

Consider now \mathbf{c}_{p^n} . Formula (3.3) shows that $\mathbf{c}_{p^n}([\tilde{Z} \rightarrow X]) = 0$ if $\text{codim } Z \neq 1, p^n$ and $\mathbf{c}_{p^n}([\tilde{Z} \rightarrow X]) = [Z]$, if $\text{codim } Z = p^n$. This proves the surjectivity of \mathbf{b} .

To check the integrality of \mathbf{b} it suffices to compute it on divisors.

Let Z be a closed subvariety of X of codimension 1, L the respective line bundle over X , and $\tilde{Z} \rightarrow Z$ a resolution of singularities of Z . Denote by $\text{pr}: \Omega^*(X) \rightarrow \text{CH}^*(X)$ the projection to the Chow theory. Then

$$\text{pr}([\tilde{Z} \rightarrow X]) - \text{pr}(c_1^\Omega(L)) = 0$$

and, thus, the element $\alpha := [\tilde{Z} \rightarrow X] - c_1^\Omega(L)$ lies in the kernel of pr , which is equal by [LM, Rem. 4.5.6] to $\mathbb{L}_{\geq 1}\Omega(X)$. Since the degree of α equals 1, α can be expressed as a linear combination with coefficients in \mathbb{L} of elements $[\tilde{Y} \rightarrow X]$, where $\tilde{Y} \rightarrow Y$ is a resolution of singularities of a subvariety Y of X of codimension bigger than 1.

Projecting α to $K(n)$ reduces the problem of integrality of \mathbf{b} to its integrality on classes of codimension bigger than 1 and to the first Chern classes of line bundles. But for classes of codimension bigger than 1 the integrality property of \mathbf{b} was shown in the first part of the proof.

Therefore it remains to prove our theorem for the first Chern classes of line bundles. But in this case it follows from Corollary 2.8, since for a line bundle L over X

$$\mathbf{c}(c_1^{K(n)}(L)) = c_1^{\text{CH}}(L) - \frac{1}{p}c_1^{\text{CH}}(L)^{p^n} + \text{higher degree terms.}$$

In particular, $\mathbf{c}_{p^n}(c_1^{K(n)}(L)) + \frac{1}{p}\mathbf{c}_1^{K(n)}(c_1^{K(n)}(L))^{p^n} = -\frac{1}{p}c_1^{\text{CH}}(L)^{p^n} + \frac{1}{p}(c_1^{\text{CH}}(L))^{p^n} = 0$ is integral.

Finally, it remains to show the additivity of the operation $\mathfrak{C} := (\mathfrak{c}_1, \dots, \mathfrak{c}_{p^n-1}, -\mathfrak{b}, 0, \dots)$. But this follows from the explicit formulae of Example 2.5:

$$\begin{aligned} \mathfrak{C}(\alpha) + \mathfrak{C}(\beta) &= (\mathfrak{c}_1(\alpha), \mathfrak{c}_2(\alpha), \dots, -\mathfrak{b}(\alpha), 0, \dots) +_W (\mathfrak{c}_1(\beta), \mathfrak{c}_2(\beta), \dots, -\mathfrak{b}(\beta), 0, \dots) \\ &= (\mathfrak{c}_1(\alpha) + \mathfrak{c}_1(\beta), \mathfrak{c}_2(\alpha) + \mathfrak{c}_2(\beta), \dots, \\ &\quad -\mathfrak{c}_{p^n}(\alpha) - \frac{1}{p}\mathfrak{c}_1(\alpha)^{p^n} - \mathfrak{c}_{p^n}(\beta) - \frac{1}{p}\mathfrak{c}_1(\beta)^{p^n} + \frac{1}{p}(\mathfrak{c}_1(\alpha)^{p^n} + \mathfrak{c}_1(\beta)^{p^n} - (\mathfrak{c}_1(\alpha) + \mathfrak{c}_1(\beta))^{p^n}), 0, \dots) \\ &= \mathfrak{C}(\alpha + \beta) \text{ for } \alpha, \beta \in K(n)(X). \end{aligned}$$

□

Corollary 3.4. *Let X be a smooth projective variety over F . Assume that X is geometrically cellular, i.e., there exists a field extension E/F such that X_E is a cellular variety, and assume that the restriction map*

$$\text{res}_{K(n)}: K(n)(X) \rightarrow K(n)(X_E)$$

is an epimorphism. Then the restriction map

$$\text{res}_{\text{CH}^i \otimes \mathbb{Z}_{(p)}}: \text{CH}^i(X) \otimes \mathbb{Z}_{(p)} \rightarrow \text{CH}^i(X_E) \otimes \mathbb{Z}_{(p)}$$

is an epimorphism for all $i \leq p^n$.

Proof. Since X_E is cellular, $\text{CH}_{\mathbb{Z}_{(p)} \subset \mathbb{Q}}(X_E) = \text{CH}(X_E) \otimes \mathbb{Z}_{(p)}$. Now the surjectivity follows from the following commutative diagram:

$$\begin{array}{ccccc} K(n)(X) & \xrightarrow{\text{res}} & K(n)(X_E) & & \\ \downarrow & & \downarrow & & \\ \text{CH}^i(X) \otimes \mathbb{Z}_{(p)} & \xrightarrow{\quad} & \text{CH}_{\mathbb{Z}_{(p)} \subset \mathbb{Q}}^i(X) & \longrightarrow & \text{CH}^i(X_E) \otimes \mathbb{Z}_{(p)} \\ & \searrow \text{res} & & & \end{array}$$

where the vertical arrows are the surjective operations \mathfrak{c}_i , $i \leq p^n - 1$, or \mathfrak{b} , which by the construction commute with restriction maps. □

Remark 3.5. The operation \mathfrak{b} can be considered as a higher analog of Chern classes. Indeed, for $K(1)$ with $p = 2$ it coincides with $c_1^2 - c_2$, where c_i are the usual Chern classes (from K^0 to CH).

In [Vi12, Thm. 6.2] Vishik classifies all additive operations $A \rightarrow B$ between two oriented cohomology theories, where A is universal for the respective formal group law. Nevertheless, the Witt vectors do not form an oriented cohomology theory (cf. [PSh06, §2, Thm. 1(2)]) and, thus, operations \mathfrak{b} do not fit directly into Vishik's context.

4. ROST MOTIVES

Starting from this section we will develop some methods to compute the Morava K -theory.

Let R_m denote the (generalized) Rost motive of a non-zero pure symbol $\alpha \in H^m(F, \mu_p^{\otimes m})$ in the category of Chow motives with $\mathbb{Z}_{(p)}$ -coefficients. By definition R_m is indecomposable and for all field extensions K/F the following conditions are equivalent:

- (1) $(R_m)_K$ is decomposable;

- (2) $(R_m)_K \simeq \bigoplus_{i=0}^{p-1} \mathbb{Z}_{(p)}(b \cdot i)$ with $b = \frac{p^{m-1}-1}{p-1}$;
- (3) $\alpha_K = 0 \in H^m(K, \mu_p^{\otimes m})$.

The fields K from this definition are called splitting fields of R_m .

The Rost motives were constructed by Rost and Voevodsky (see [Ro06], [Vo11]). Namely, for all pure symbols α there exists a smooth projective ν_{m-1} -variety X (depending on α) over F such that the Chow motive of X has a direct summand isomorphic to R_m . The variety X is called a *norm variety* of α .

E.g., if $p = 2$ and $\alpha = (a_1) \cup \dots \cup (a_m)$ with $a_i \in F^\times$, then one can take for X the projective quadric given by the equation $\langle\langle a_1, \dots, a_{m-1} \rangle\rangle \perp \langle -a_m \rangle = 0$, where $\langle\langle a_1, \dots, a_{m-1} \rangle\rangle$ denotes the Pfister form.

By [ViYa07, Sec. 2] there is a unique lift of the Rost motive R_m to the category of Ω -motives and, since Ω is the universal oriented cohomology theory, there is a well-defined Rost motive in the category of A^* -motives for any oriented cohomology theory A^* . We will denote this A -motive by the same letter R_m . By $\mathbb{T}(l)$, $l \geq 0$, we denote the Tate motives in the category of A -motives. If $A = \mathrm{CH} \otimes \mathbb{Z}_{(p)}$, we keep the usual notation $\mathbb{T}(l) = \mathbb{Z}_{(p)}(l)$.

Proposition 4.1. *Let p be a prime number, n and m be natural numbers and $b = \frac{p^{m-1}-1}{p-1}$. For a non-zero pure symbol $\alpha \in H^m(F, \mu_p^{\otimes m})$ consider the respective Rost motive R_m . Then*

- (1) *If $n < m - 1$, then the $K(n)$ -motive R_m is a sum of p Tate motives $\bigoplus_{i=0}^{p-1} \mathbb{T}(b \cdot i)$.*
- (2) *If $n = m - 1$, then the $K(n)$ -motive R_m is a sum of the Tate motive \mathbb{T} and an indecomposable motive L such that*

$$K(n)(L) \simeq (\mathbb{Z}^{\oplus(p-1)} \oplus (\mathbb{Z}/p)^{\oplus(m-2)(p-1)}) \otimes \mathbb{Z}_{(p)}[v_n, v_n^{-1}].$$

For a field extension K/F the motive L_K is isomorphic to a direct sum of twisted Tate motives iff it is decomposable and iff the symbol $\alpha_K = 0$.

- (3) *If $n > m - 1$, then the $K(n)$ -motive R_m is indecomposable and its realization equals the group $\mathrm{CH}(R_m) \otimes \mathbb{Z}_{(p)}[v_n, v_n^{-1}]$. For a field extension K/F the motive $(R_m)_K$ is decomposable iff $\alpha_K = 0$. In this case $(R_m)_K$ is a sum of p Tate motives.*

Proof. Denote by \overline{R}_m the scalar extension of R_m to its splitting field. By [Ya12, Prop. 11.11] (cf. [ViYa07, Thm. 3.5, Prop. 4.4]) the restriction map for the BP -theory

$$(4.2) \quad \mathrm{res}: BP(R_m) \rightarrow BP(\overline{R}_m) = BP(\mathrm{Spec} F)^{\oplus p}$$

is injective, and the image equals

$$(4.3) \quad BP(R_m) \simeq BP(\mathrm{Spec} F) \oplus I(p, m-2)^{\oplus(p-1)},$$

where $I(p, m-2)$ is the ideal in the ring $BP(\mathrm{Spec} F) = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ generated by the elements $\{p, v_1, \dots, v_{m-2}\}$.

(1) Assume first that $n < m - 1$. Since the ideal $I(p, m-2)$ contains v_n for $n < m - 1$ and v_n is invertible in $K(n)(\mathrm{Spec} F)$, we immediately get that all elements in $K(n)(\overline{R}_m)$ are rational, i.e., are defined over the base field. By the properties of the Rost motives

$$(4.4) \quad \Omega^l(R_m \times R_m) = \bigoplus_{i+j=l} \Omega^i(R_m) \otimes \Omega^j(\overline{R}_m)$$

for all l . Since Ω is a universal theory, the same formula holds for BP and for $K(n)$. Therefore all elements in $K(n)(\overline{R}_m \times \overline{R}_m)$ are rational, and this gives the first statement of the proposition.

(2) Assume now that $n = m - 1$. Let X be a norm variety for the symbol α . In particular, $\dim X = p^{m-1} - 1 = p^n - 1$. Since the Morava-Euler characteristic of X equals $u \cdot v_n$ for some $u \in \mathbb{Z}_{(p)}^\times$ (see Section 2.9), the element $v_n^{-1} \cdot u^{-1}(1 \times 1) \in K(n)(X \times X)$ is a projector defining the Tate motive \mathbb{T} . Thus, we get the decomposition $R_m \simeq \mathbb{T} \oplus L$ for some motive L . We claim that L is indecomposable.

Indeed, by [LM, Thm. 4.4.7] the elements of $K(n)^{p^n-1}(R_m \times R_m)$ are linear combinations of elements of the form $v_n^s \cdot [Y \rightarrow X \times X]$, where Y is a resolution of singularities of a subvariety of $X \times X$, and $-s(p^n - 1) + \text{codim } Y = p^n - 1$. In particular, $s = 0, 1, -1$ and $\text{codim } Y = 0, p^n - 1, 2(p^n - 1)$. By formula (4.4) and by the injectivity of the restriction map for BP , it follows that there are at most three rational projectors in $K(n)(\overline{R}_m \times \overline{R}_m)$. These are the diagonal, the projector $v_n^{-1} \cdot u^{-1}(1 \times 1)$ constructed above and their difference (which defines the motive L). Therefore by Rost nilpotency (see Section 2.10) the motive L is indecomposable over F .

Taking the tensor product $- \otimes_{BP(\text{Spec } F)} K(n)(\text{Spec } F)$ with formula (4.2) and using (4.3) one immediately gets the formula for $K(n)(L)$.

(3) The same arguments show that R_m is indecomposable for the Morava K -theory $K(n)$ for $n > m - 1$, cf. [Nes13, Ex. 6.14]. \square

Remark 4.5. This proposition demonstrates a difference between K^0 and the Morava $K(n)$ -theory, when $n > 1$. By [Pa94] K^0 of all twisted flag varieties is \mathbb{Z} -torsion-free. This is not the case for $K(n)$, $n > 1$.

Moreover, the same arguments as in the proof of the proposition show that the connective K -theory $CK(1)$ (see [Cai08]) of Rost motives R_m for $m > 2$ contains non-trivial \mathbb{Z} -torsion.

Remark 4.6. The Chow groups of the Rost motives are known; see [KM02, Thm. 8.1], [KM13, Thm. RM.10], [Ya12, Cor. 10.8], [Vi07, Section 4.1].

5. GEOMETRIC FILTRATION ON A PRODUCT OF QUADRICS

Starting from this section we will introduce a method to decompose the A -motive of a projective quadric for an arbitrary oriented cohomology theory A . We remark that it is not sufficient to assume that $A = \Omega$ is universal. E.g., the Ω -motive of a generic quadric is indecomposable (since due to Vishik the Chow motive of a generic quadric is indecomposable (see [Ka12, Thms. 3.1, 4.1]) and by [ViYa07, Sec. 2] this implies that the Ω -motive of a generic quadric is indecomposable). On the other hand, the K^0 -motives of all quadrics are always decomposable (see [Pa94]).

Let q be a regular quadratic form over a field F and $Q = \{\langle u \rangle \mid q(u) = 0\}$ be the respective n -dimensional smooth projective quadric. Define

$$X = \left\{ (\langle u \rangle, \langle v \rangle) \in Q \times Q \mid q(u, v) = 0 \right\}.$$

There is the following filtration of $Q \times Q$.

$$Q \times Q \supset X \supset Q,$$

where $Q \subset X$ is the diagonal embedding.

The projection to the first component

$$p_1: (Q \times Q) \setminus X \rightarrow Q$$

is an \mathbb{A}^n -fibration. Let $\mathrm{OGr}(1, 2, Q)$ denote the Grassmannian of isotropic flags of subspaces of dimensions 1 and 2. The map

$$\begin{aligned} X \setminus Q &\rightarrow \mathrm{OGr}(1, 2, Q) \\ (\langle u \rangle, \langle v \rangle) &\mapsto (\langle u \rangle \leq \langle u, v \rangle) \end{aligned}$$

is an \mathbb{A}^1 -fibration.

Let τ_2 be the tautological vector bundle over $\mathrm{OGr}(1, 2, Q)$ of rank 2. Consider

$$\begin{aligned} \mathrm{OGr}(1, 2, Q) &\xleftarrow{\pi} \mathbb{P}(\tau_2) \xrightarrow{f} Q \times Q \\ (\langle u \rangle \leq \langle u, v \rangle) &\mapsto (\langle u \rangle \leq \langle u, v \rangle, \langle w \rangle \leq \langle u, v \rangle) \mapsto (\langle u \rangle, \langle w \rangle) \end{aligned}$$

Note $f(\mathbb{P}(\tau_2)) = X$ and so f is a resolution of singularities of X .

By [NeZ06, Proof of Thm. 4.4] we have

$$\begin{aligned} (5.1) \quad A^*(Q \times Q) &\xleftarrow{\sim} A^*(Q) \oplus A^{*-1}(\mathrm{OGr}(1, 2, Q)) \oplus A^{*-\dim Q}(Q) \\ p_1^*(x) + f_* \circ \pi^*(y) + i_*(z) &\mapsto x \oplus y \oplus z \end{aligned}$$

where $i: Q \rightarrow Q \times Q$ is the diagonal embedding.

6. EQUIVARIANT THEORIES AND MORPHISMS

In this section we give general information about *equivariant* oriented cohomology theories following Brion and Krishna (see [Bri97], [Kr10] and references there).

Let T be a split torus (i.e. a product of several copies of the multiplicative group \mathbb{G}_m) and X a smooth projective T -variety, where T acts on X with a finite number of fixed points.

Let A_T^* be an oriented T -equivariant cohomology theory which is obtained from the equivariant cobordism theory Ω_T^* by a change of the coefficients. Then there is an injective ring homomorphism

$$(6.1) \quad A_T^*(X) \hookrightarrow \bigoplus_{x \in X^T} A_T^*(x)$$

induced by the embedding of these fixed points $X^T \subset X$, see [Kr10, Cor. 7.2].

6.2 (pull-back). Let X and Y be smooth projective T -varieties with finite number of fixed points and let $g: X \rightarrow Y$ be a T -equivariant morphism of relative dimension d .

Then we have a commutative diagram

$$\begin{array}{ccc} A_T^*(Y) & \xrightarrow{g^*} & A_T^*(X) \\ \downarrow & & \downarrow \\ \bigoplus_{y \in Y^T} A_T^*(y) & \xrightarrow{\tilde{g}^*} & \bigoplus_{x \in X^T} A_T^*(x) \end{array}$$

where $\tilde{g}^*(a)_x = a_{g(x)}$, $x \in X^T$, $a \in \bigoplus_{y \in Y^T} A_T^*(y)$.

6.3 (push-forward). For every $x \in X^T$ the torus T acts on the vector space $T_{X,x}$, where $T_{X,x}$ denotes the tangent bundle of X at the point x . We assume that this representation of T has no zero weights and define $c_{top}^T(T_{X,x})$ as the product of all its weights with multiplicities. Furthermore, we assume that $c_{top}^T(T_{Y,y}) \neq 0$ for all $y \in Y^T$.

Then there is a commutative diagram

$$\begin{array}{ccc} A_T^*(X) & \xrightarrow{g^*} & A_T^{*-d}(Y) \\ \downarrow & & \downarrow \\ \bigoplus_{x \in X^T} A_T^*(x) & \xrightarrow{\tilde{g}_*} & \bigoplus_{y \in Y^T} A_T^{*-d}(y) \end{array}$$

where

$$\tilde{g}_*(a)_y = \sum_{f(x)=y} a_x \cdot \frac{c_{top}^T(T_{Y,y})}{c_{top}^T(T_{X,x})},$$

$y \in Y^T$, $a \in \bigoplus_{x \in X^T} A_T^*(x)$.

Let now $y \in Y^T$ and consider the Cartesian square

$$\begin{array}{ccc} Z & \xrightarrow{j} & X \\ \downarrow \zeta & & \downarrow g \\ y & \xrightarrow{\zeta} & Y \end{array}$$

Assume that the fiber $Z := g^{-1}(y)$ is reduced and smooth. Then

$$(6.4) \quad \tilde{g}_*(a)_y = \zeta_*(j^*(a) \cdot c_{top}^T(\frac{T_{Y,y}}{d_x g(T_{X,x})}))$$

for all $a \in \bigoplus_{x \in X^T} A_T^*(x)$.

7. EQUIVARIANT COMPUTATIONS

Now we apply the previous considerations to the geometric filtration on $Q \times Q$ constructed in Section 5.

7.1 (Description of the method). Let Q be an arbitrary smooth projective quadric over F . Our goal is to find a decomposition of the A -motive of Q , i.e., to construct projectors in $A(Q \times Q)$.

By Rost nilpotency it suffices to construct rational projectors in $A(\overline{Q} \times \overline{Q})$, where \overline{Q} denotes the extension of scalars to a splitting field of the quadric Q .

To construct rational projectors we use formula (5.1) which is compatible with scalar extensions. Namely, we start with rational elements in $A(\overline{Q})$ and $A(\text{OGr}(1, 2, \overline{Q}))$ (e.g. with Chern classes of rational bundles), and compute their images in $A(\overline{Q} \times \overline{Q})$ under the map of formula (5.1).

To compute these images we use the T -equivariant theory A_T , the injection (6.1) and the concrete formulas for pullbacks and pushforwards for A_T described in Section 6. In particular, we need to know how the fixed points on \overline{Q} , $\text{OGr}(1, 2, \overline{Q})$ and $\mathbb{P}(\tau_2)$ look like and we need a description of the weights of the tangent bundles at the fixed points. Below we will give this explicit description.

This reduces the computation of projectors in $A(Q \times Q)$ to combinatorics, which can be performed on a computer.

Let (V, q) be a $2l$ -dimensional *split* quadratic space with basis

$$\{e_1, \dots, e_l, e_{-l}, \dots, e_{-1}\}$$

and $q(e_i) = 0$, $q(e_i + e_j) = \delta_{i,-j}$. Let Q be the corresponding projective split quadric and $G = O^+(Q)$ be the respective orthogonal group with a split maximal torus T such that $\langle e_i \rangle$ is the weight subspace of V of weight χ_i , i.e., $te_i = \chi_i(t)e_i$ for all $t \in T$, $i = 1, \dots, l$. The character group of T is $\mathbb{Z}\chi_1 \oplus \dots \oplus \mathbb{Z}\chi_l$.

The torus T acts naturally on Q , $OGr(1, 2, Q)$, and $\mathbb{P}(\tau_2)$ with finite number of fixed points. The group G acts on these varieties, so the Weyl group W of G acts on the respective fixed points. If x is a fixed point and the weights of $T_{X,x}$ are $\rho_1, \dots, \rho_{\dim X}$ (with multiplicities), then the weights of $T_{X,w(x)}$ are $w(\rho_1), \dots, w(\rho_{\dim X})$, $w \in W$. So, it is sufficient to compute the weights for representatives in W -orbits of fixed points. For $X = G/P$, where P is a parabolic subgroup of G , the weights at the point $1 \cdot P$ are the roots of the unipotent radical of P^- .

We have a commutative diagram

$$\begin{array}{ccc} A_T^*(Q) \oplus A_T^{*-1}(OGr(1, 2, Q)) \oplus A_T^{*-\dim Q}(Q) & \xrightarrow{\simeq} & A_T^*(Q \times Q) \\ \downarrow & & \downarrow \\ \bigoplus_{x \in Q^T} A_T^*(x) \oplus \bigoplus_{y \in OGr(1, 2, Q)^T} A_T^{*-1}(y) \oplus \bigoplus_{x' \in Q^T} A_T^{*-\dim Q}(x') & \longrightarrow & \bigoplus_{z \in (Q \times Q)^T} A_T^*(z) \end{array}$$

7.2 (Fixed points on a quadric). The fixed points on Q are the lines $\langle w(e_1) \rangle$, $w \in W$, i.e., the lines $\langle e_i \rangle$, $i = -l, \dots, -1, 1, \dots, l$.

The weights at $\langle e_1 \rangle$ are $\pm\chi_2 - \chi_1, \dots, \pm\chi_l - \chi_1$.

7.3 (Fixed points on $OGr(1, 2, Q)$). The fixed points on Q are the flags

$$(\langle w(e_1) \rangle \leq \langle w(e_1), w(e_2) \rangle), \quad w \in W.$$

The weights at $(\langle e_1 \rangle \leq \langle e_1, e_2 \rangle)$ are

$$\pm\chi_2 - \chi_1, \pm\chi_3 - \chi_1, \dots, \pm\chi_l - \chi_1, \pm\chi_3 - \chi_2, \pm\chi_4 - \chi_2, \dots, \pm\chi_l - \chi_2.$$

7.4 (Fixed points on $\mathbb{P}(\tau_2)$). There are two W -orbits of fixed points on $\mathbb{P}(\tau_2)$:

$$(\langle e_1 \rangle \leq \langle e_1, e_2 \rangle, \langle e_1 \rangle \leq \langle e_1, e_2 \rangle) \text{ and } (\langle e_1 \rangle \leq \langle e_1, e_2 \rangle, \langle e_2 \rangle \leq \langle e_1, e_2 \rangle).$$

The weights at the first (resp. at the second) of these points are the same as for $OGr(1, 2, Q)$ at $(\langle e_1 \rangle \leq \langle e_1, e_2 \rangle)$ together with the weight of \mathbb{P}^1 at $\langle e_1 \rangle$ (resp. at $\langle e_2 \rangle$), which is $\chi_2 - \chi_1$ (resp. $\chi_1 - \chi_2$).

$$\begin{array}{ccccc}
A_T^{*-1}(\mathrm{OGr}(1, 2, Q)) & \xrightarrow{\pi^*} & A_T^{*-1}(\mathbb{P}(\tau_2)) & \xrightarrow{f^*} & A_T^*(Q \times Q) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{y \in \mathrm{OGr}(1, 2, Q)^T} A_T^{*-1}(y) & \longrightarrow & \bigoplus_{u \in \mathbb{P}(\tau_2)^T} A_T^{*-1}(u) & \longrightarrow & \bigoplus_{z \in (Q \times Q)^T} A_T^*(z)
\end{array}$$

The map $\pi: \mathbb{P}(\tau_2) \rightarrow \mathrm{OGr}(1, 2, Q)$ is on the fixed points 2 : 1 and the points

$$(\langle e_1 \rangle \leq \langle e_1, e_2 \rangle, \langle e_1 \rangle \leq \langle e_1, e_2 \rangle) \text{ and } (\langle e_1 \rangle \leq \langle e_1, e_2 \rangle, \langle e_2 \rangle \leq \langle e_1, e_2 \rangle)$$

map to $\langle e_1 \rangle \leq \langle e_1, e_2 \rangle$.

Then the map f maps

$$(\langle e_1 \rangle \leq \langle e_1, e_2 \rangle, \langle e_1 \rangle \leq \langle e_1, e_2 \rangle) \text{ to } (\langle e_1 \rangle, \langle e_1 \rangle)$$

and

$$(\langle e_1 \rangle \leq \langle e_1, e_2 \rangle, \langle e_2 \rangle \leq \langle e_1, e_2 \rangle) \text{ to } (\langle e_1 \rangle, \langle e_2 \rangle).$$

So, the point $(\langle e_1 \rangle, \langle e_1 \rangle) \in (Q \times Q)^T$ has $2l-2$ preimages and $(\langle e_1 \rangle, \langle e_2 \rangle)$ has one preimage.

By formula (6.4) applied to f and $z := (\langle e_1 \rangle, \langle e_2 \rangle)$ we obtain

$$\tilde{f}_*(a)_z = a_u \cdot (-\chi_1 - \chi_2),$$

where $u = (\langle e_1 \rangle \leq \langle e_1, e_2 \rangle, \langle e_2 \rangle \leq \langle e_1, e_2 \rangle)$.

Applying $w \in W$ we cover all coordinates of $\tilde{f}_*(a)$ in the W -orbit of $(\langle e_1 \rangle, \langle e_2 \rangle)$. Note that the fixed points in the orbit of $(\langle e_1 \rangle, \langle e_{-1} \rangle)$ have no preimages under f . So, these coordinates of $\tilde{f}_*(a)$ equal 0.

It remains to cover coordinates in the orbit of $z := (\langle e_1 \rangle, \langle e_1 \rangle) \in (Q \times Q)^T$. We have $f^{-1}(\langle e_1 \rangle, \langle e_1 \rangle)$ is isomorphic to the $(2l-2)$ -dimensional subquadric Z of Q given by the restriction of q to $\langle e_2, \dots, e_l, e_{-l}, \dots, e_{-2} \rangle$. The embedding $j: Z \rightarrow \mathbb{P}(\tau_2)$ is given by $j(\langle v \rangle) = (\langle e_1 \rangle \leq \langle e_1, v \rangle, \langle e_1 \rangle \leq \langle e_1, v \rangle)$. By formula (6.4)

$$\tilde{f}_*(a)_z = \zeta_*(j^*(a) \cdot (-\chi_1 - \chi_2) \cdot \prod_{i=3}^l (\chi_i - \chi_1)(-\chi_i - \chi_1)).$$

This finishes the description of the fixed points, weights and push-forwards needed to apply our method 7.1.

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